Math 249 Lecture 8 Notes

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1 Projections Onto Representations

1.1 The character basis

We review a point made in the proof of the orthogonality of characters theorem from last lecture. Why is the number of conjugacy classes $\leq$ the number of irreducible representations?

$\mathbb{C}G \circlearrowright \operatorname{Hom}(V_i, V_i)$ for each irreducible representation V_i, so we get a map $\mathbb{C}G \to \bigoplus_i \operatorname{Hom}(V_i, V_i)$. This map is injective because it has 0 kernel; suppose that $\rho_i(x) = 0$ for every i. Then left multiplication by x is 0 in $\mathbb{C}G$, so $x = 0$. Then for $z \in Z(\mathbb{C}G)$, $\rho_i(z) : V_i \to V_i$ is a G-module homomorphism because it commutes with every $\varphi \in \operatorname{Hom}(V_i, V_i)$. So each $\rho_i(z) = c_i I_{V_i}$ by Schur's lemma. This implies that $\dim(Z(\mathbb{C}G)) \leq$ the number of irreducible representations. For a conjugacy class C, let $\delta_C = \sum_{g \in X} g$; this is a basis for the class of functions constant on conjugacy classes. Each $\delta_C \in Z(\mathbb{C}G)$ because

$$h \left(\sum_{g \in C} g \right) h^{-1} = \sum_{g \in C} hgh^{-1} = \sum_{g \in C} g,$$

where h just reindexes the elements in the sum. So the number of conjugacy classes $\leq \dim Z(\mathbb{C}G)$.

1.2 Projections

What we get from the above is that $Z(\mathbb{C}G) \cong \bigoplus_i \mathbb{C} \cdot I_{V_i}$. Then for each irreducible V_i, we can find an element $e_i \in Z(\mathbb{C}G)$ such that $\rho_j(e_i) = \delta_{i,j} I_{V_j}$,; moreover, $e_1^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. Let $V^{(i)} = \bigoplus_j W_j$, where the index j ranges over all $W_j \cong V_i$. then e_j acts on V as a projection onto $V^{(i)}$. The $V^{(i)}$ are also unique.

Example 1.1. Let R be the Reynolds operator (an element of $\mathbb{C}G$)

$$R = \frac{1}{|G|} \sum_{g \in G} g.$$

Then $R = e_{\mathbb{1}}$, the projection onto the trivial part of the representation.

2 Irreducible character tables

2.1 Hermitian character tables

Recall the character table, introduced last lecture. The character table with only irreducible representations will be a square matrix because the number of irreducible representations is equal to the number of conjugacy classes of G. Since the characters are orthonormal, we can rescale the columns to make the rows orthogonal. This is the matrix with elements $\sqrt{|C_j| \,/\, |G|}\chi_i(g_j)$. Compare this with the original character table matrix, which had entries $\chi_i(g_j)$. This matrix is a Hermitian matrix ($A^{-1} = A^*$). The columns are orthonormal because

$$\sqrt{\frac{|C_j|}{|G|}}\sqrt{\frac{|C_k|}{|G|}}\sum_i \chi_i(g_j)\overline{\chi_i(g_k)} = (A^*A)_{k,j} = (I)_{k,j} = \delta_{k,j},$$

When $k = j$, this gives us that $(|G| \,/\, |C_j|)\sum_i |\chi_i(g_j)|^2 = 1$, so

$$\sum_i |\chi_i(g_j)|^2 = \frac{|G|}{|C_j|}.$$

And in the case of the the conjugacy class of the identity e, we have

$$\sum_i |\chi_i(e)|^2 = |G|\,,$$

a nice expression for the order of a group.

2.2 The irreducible character table of S_4

We can use all these facts we've proved to help us figure out the character table of a group.

Example 2.1. The irreducible character table of S_4 is

S_4	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$1\,2\,3\,4)$
$\chi_{\mathbb{1}} = \chi_{\yng(4)}$	1	1	1	1	1
$\chi_{\yng(3,1)}$	3	1	-1	0	1
$\chi_{\yng(2,2)}$	2	0	2	-1	0
$\chi_{\yng(2,1,1)}$	3	-1	-1	0	1
$\chi_{\varepsilon} = \chi_{\yng(1,1,1,1)}$	1	-1	1	1	-1

What representation does $\chi_{2,2}$ correspond to? The character table can help us find the normal subgroups of a group. The character table on a factor group will be contained in the character table (by deleting rows and columns), and any row where some element is equal to the leftmost element ($\chi_i(e)$) indicates that the union of those conjugacy classes is a normal subgroup of G. In the third row of the above table, the first and third column share the number 2. $\{e\} \cup \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup, so there is a homomorphism S_4 to some nonabelian 6-element group. The only such group is S_3, and this homomorphism is action by conjugation on the conjugacy class $C_{2,2}$. So the representation is $S_4 \mapsto S_3 \circlearrowright \mathbb{C}^3 / \langle (1,1,1) \rangle$.